Ordinary and Partial Differential Equations

Maple is the world leader in finding exact solutions to ordinary and partial differential equations. Maple 2021 extends that lead even further with new algorithms and techniques for solving more ODEs and PDEs.

For Maple 2021, there are significant improvements in dsolve for the exact solution of 2^{nd} order linear ODEs using hypergeometric functions. The algorithms implemented are at the frontier of the understanding of this problem, and handle classes of extended equations with apparent singularities as well as the most common linear equations with 4 and 5 singularities.

For ODEs and PDEs, the LieAlgebrasOfVectorFields package in Maple 2021 has a new command MapDE , for analyzing the possible linearization of polynomially nonlinear equations, and determining the corresponding mapping when it exists, allowing in that way for the computation of more approximate and exact solutions.

The new Student[ODEs] package covers the material in a standard first course in ODEs and provides step-by-step tools for solving ODEs as well as interactive visualization. For details, see Student Packages.

Advanced type of Hypergeometric solutions for 2^{nd} order linear ODEs

When they exist, 2F1, 1F1 and 0F1 hypergeometric solutions for 2^{nd} order linear ODEs are now computable in a rather general case. An equation with 3 regular singularities and any number of apparent singularities can now be solved in terms of 2F1 equations using the algorithms presented in (6) and (7). Likewise, an equation with 2 or 1 singularities, one of which is irregular, can now be solved in general using 1F1 and 0F1 hypergeometric functions using the algorithms presented in (1), (2) and (3). Common equations with up to 5 regularities (plus any number of apparent singularities) can now be solved using mappings and the tables from references (4) and (5). All this significantly extends the previous capabilities in solving linear equations, and also of higher order and nonlinear ODE and PDE that through a reduction or symmetry process require the solving of intermediate 2^{nd} order linear ODEs.

The new algorithms are now automatically used by *dsolve* and also accessible through a new DEtools command, *hypergeometricsols*.

Examples

Despite the powerful algorithms for hypergeometric solutions of previous Maple releases, none of the following examples could be solved. Now they can. This equation admits 2F1 solutions where the argument is polynomial in *x* of degree 3

> $ode_1 := \frac{d^2}{dx^2}\, y(x) = \frac{1}{432\left(x-\frac{2}{3}\right)\left(1+b\,(x-1)\,x^2\right)(x-1)}\left(\left(1296\left(\left(a-\frac{5}{6}\right)x^2+\left(-\frac{4\,a}{3}+\frac{10}{9}\right)x+\frac{4\,a}{9}-\frac{4}{9}\right)x\,(x-1)\,b+216\,x-288\right)\right)d\,/\,(dx)\;y(x) - \frac{9\left(a-\frac{5}{6}\right)\left(a-\frac{1}{6}\right)\left(x-\frac{2}{3}\right)^2 b}{4\left(1+b\,(x-1)\,x^2\right)(x-1)}\,y(x):$

> $dsolve(ode_1)$

$$y(x) = \_C1\;{}_2F_1\left(\frac{1}{12}-\frac{a}{2},-\frac{a}{2}+\frac{5}{12};\frac{1}{2};-b\,(x-1)\,x^2\right)+\_C2\,x\sqrt{1-x}\;{}_2F_1\left(\frac{7}{12}-\frac{a}{2},\frac{11}{12}-\frac{a}{2};\frac{3}{2};-b\,(x-1)\,x^2\right) \quad \textbf{(1.1)}$$

Verify this solution

> $odetest(\textbf{(1.1)}, ode_1)$

$$0 \quad \textbf{(1.2)}$$

This other equation admits solutions in terms of modified Bessel functions of the first and second kind

> $ode_2 := \frac{d^2}{dx^2}\, y(x) = \frac{(8\,x-48)}{(2\,x-9)\,(2\,x-15)}\,\frac{d}{dx}\,y(x) + \frac{\left(4\,x^6-96\,x^5+931\,x^4-4452\,x^3+10539\,x^2-12474\,x+9720\right)}{4\,x^2\,(x-3)^2\,(2\,x-9)\,(2\,x-15)}\,y(x):$

> $dsolve(ode_2)$

$$y(x) = \frac{1}{x-3}\left(\_C1\,x\left(\left(4\,x^3-36\,x^2+141\,x-225\right)I_1\left(\frac{x}{2}\right)+\left(-4\,x^3+24\,x^2-63\,x+27\right)I_2\left(\frac{x}{2}\right)\right)\right) \quad \textbf{(1.3)}$$

$$+\frac{1}{x-3}\left(\_C2\, x\left(\left(-4x^3+36x^2-141x+225\right)K_1\left(\frac{x}{2}\right)+\left(-4x^3+24x^2-63x+27\right)K_2\left(\frac{x}{2}\right)\right)\right)$$

Moreover, note that each independent solution involves a *linear combination* of functions. Indeed, while Bessel (or more generally, 0F1) solutions are related to equations with 1 singularity of irregular kind, this example has other *regular* singularities:

> $DEtools[singularities](ode_2)$

$$regular=\left\{0, 3, \frac{9}{2}, \frac{15}{2}\right\}, irregular=\{\infty\} \qquad \textbf{(1.4)}$$

thus requiring the use of a linear combination of pFq functions to construct a solution. A similar situation happens with the next example where, due to the presence of apparent singularities, a linear combination of - this time 1F1 - hypergeometric functions (KummerM and KummerU) is required to solve the problem

> $ode_3 := \frac{d^2}{dx^2}y(x)=\frac{\left(16x^4+32x^2-36x+8\right)}{16x^4-16x^3+32x^2-4x}\frac{d}{dx}y(x)-\frac{12\left(x^3+\frac{1}{3}x^2+x-\frac{9}{4}\right)}{16x^4-16x^3+32x^2-4x}y(x):$

> $dsolve(ode_3)$

$$y(x)=\_C1\left(12\left(x-\frac{1}{2}\right)^2 M\left(\frac{5}{4}, 3, x\right)+(-20x+5)\,M\left(\frac{9}{4}, 3, x\right)\right)+\_C2\left(16\left(x-\frac{1}{2}\right)^2 U\left(\frac{5}{4}, 3, x\right)+(20x-5)\,U\left(\frac{9}{4}, 3, x\right)\right) \qquad \textbf{(1.5)}$$

In addition to computing new solutions out of reach in previous releases, when the linear ODE is of Heun type, so it has 4 regular singularities or one of its confluent cases, and it happens to be one of the special Heun function cases that can be expressible using hypergeometric functions, *both kinds of solutions* are now computable. This equation is of the Heun triconfluent type

> $ode_4 := \frac{d^2}{dx^2}y(x)=\left(4x^4-4x^2+10x-2\right)y(x):$

The algorithms in dsolve perceive that and solve it accordingly

> $dsolve(ode_4)$

$$y(x)=\_C1\, e^{-\frac{2}{3}x^3+x}\, HT\left(\frac{3\, 6^{2/3}}{4}, -\frac{15}{2}, -6^{1/3}, \frac{x\, 6^{2/3}}{3}\right)+\_C2\, e^{\frac{2}{3}x^3-x}\, HT\left(\frac{3\, 6^{2/3}}{4}, \right. \qquad \textbf{(1.6)}$$

$$\frac{15}{2}, -6^{1/3}, -\frac{x\,6^{2/3}}{3}\Big)$$

By using the option of indicating the method, a solution in terms of *linear combinations* of Airy functions (of the 0F1) is now also computable

> $dsolve\left(ode_4, [hypergeometricsols]\right)$

$$y(x) = \_C1\,\left((2x+1)\,Ai'(x^2-1) + (2x^2+x-1)\,Ai(x^2-1)\right) + \_C2\,\left((2x+1)\,Bi'(x^2-1) + (2x^2+x-1)\,Bi(x^2-1)\right) \quad \textbf{(1.7)}$$

To express this solution in terms of the more general 0F1 form you can use

> $convert(\textbf{(1.7)}, \text{hypergeom})$

$$y(x) = \_C1\left(\frac{1}{6\,\Gamma\left(\frac{2}{3}\right)}\left((2x+1)\left(-\frac{3\,3^{1/6}\,\Gamma\left(\frac{2}{3}\right)^2\,{}_0F_1\left(;\frac{1}{3};-\frac{(-x^2+1)^3}{9}\right)}{\pi} + (x^2-1)^2\,3^{1/3}\,{}_0F_1\left(;\frac{5}{3};-\frac{(-x^2+1)^3}{9}\right)\right)\right) + (2x^2+x-1)\left(\frac{3^{1/3}\,{}_0F_1\left(;\frac{2}{3};-\frac{(-x^2+1)^3}{9}\right)}{3\,\Gamma\left(\frac{2}{3}\right)} - \frac{(x^2-1)\,3^{1/6}\,\Gamma\left(\frac{2}{3}\right)\,{}_0F_1\left(;\frac{4}{3};-\frac{(-x^2+1)^3}{9}\right)}{2\,\pi}\right)\right) + \_C2\left(\frac{1}{(6x^2-6)\,\Gamma\left(\frac{2}{3}\right)}\left((2x+1)\left(\frac{3\,3^{2/3}\,\Gamma\left(\frac{2}{3}\right)^2\,{}_0F_1\left(;\frac{1}{3};\frac{(x^2-1)^3}{9}\right)}{\pi} + 3^{5/6}\,(x-1)^2\,(x+1)^2\,{}_0F_1\left(;\frac{5}{3};\frac{(x^2-1)^3}{9}\right)\right)(x-1)\,(x+1)\right) + (2x^2+x \quad \textbf{(1.8)}$$

$$- 1)\left(\frac{3^{5/6}\, {}_0F_1\left(;\frac{2}{3};-\frac{(-x^2+1)^3}{9}\right)}{3\,\Gamma\left(\frac{2}{3}\right)} + \frac{3^{2/3}\,(x^2-1)\,\Gamma\left(\frac{2}{3}\right)\, {}_0F_1\left(;\frac{4}{3};-\frac{(-x^2+1)^3}{9}\right)}{2\,\pi}\right)\Bigg)\Bigg)$$

A similar situation, this time with regards to this equation of the Heun biconfluent class

> $ode_5 := \frac{d^2}{dx^2}\, y(x) = -\frac{\left(x^2+3\,x+4\right)}{3\,x}\,\frac{d}{dx}\, y(x) + \frac{(x+1)}{9\,x}\, y(x) :$

> $dsolve\left(ode_5\right)$

$$y(x) = \_C1\ HB\left(\frac{1}{3}, \sqrt{6}, -3, \frac{14\sqrt{6}}{9}, \frac{\sqrt{6}\,x}{6}\right) e^{-\frac{x\,(x+6)}{6}} + \frac{\_C2\ HB\left(-\frac{1}{3}, \sqrt{6}, -3, \frac{14\sqrt{6}}{9}, \frac{\sqrt{6}\,x}{6}\right) e^{-\frac{x\,(x+6)}{6}}}{x^{1/3}} \tag{1.9}$$

The solution is expressible in terms of linear combinations of modified Bessel functions of the 1st and 2nd kinds

> $dsolve\left(ode_5, [hypergeometricsols]\right)$

$$y(x) = \frac{1}{\sqrt{4+x}\,x^{1/6}}\left(\_C1\ e^{-\frac{x\,(x+6)}{12}}\left(\left(x^2+6\,x+16\right) I_{\frac{1}{3}}\left(\frac{(4+x)\sqrt{(4+x)\,x}}{12}\right) + (4+x)\sqrt{(4+x)\,x}\; I_{\frac{4}{3}}\left(\frac{(4+x)\sqrt{(4+x)\,x}}{12}\right)\right)\right) + \frac{1}{\sqrt{4+x}\,x^{1/6}}\left(\_C2\ e^{-\frac{x\,(x+6)}{12}}\left(\left(x^2+6\,x+16\right) K_{\frac{1}{3}}\left(\frac{(4+x)\sqrt{(4+x)\,x}}{12}\right) - (4+x)\sqrt{(4+x)\,x}\; K_{\frac{4}{3}}\left(\frac{(4+x)\sqrt{(4+x)\,x}}{12}\right)\right)\right) \tag{1.10}$$

The following example is of the same kind but more general: it belongs to the Heun general class, but again it is one of those special cases where pFq function solutions exist, in this case 2F1 with rational coefficients

> $ode_6 := \frac{d^2}{dx^2}\, y(x) = -\frac{3\,(45\,x^2+14\,x+1)}{4\,x\,(27\,x^2+14\,x+3)}\,\frac{d}{dx}\,y(x) + \frac{3\,(x+1)^2\,(b-3)\,(b+3)}{64\,x^2\,(27\,x^2+14\,x+3)}\,y(x):$

> $dsolve(ode_6)$

$$y(x) = \frac{1}{(27\,x^2+14\,x+3)^{1/4}} \Bigg(\_C1\, HG\Bigg(\frac{-4\,I\sqrt{2}+7}{4\,I\sqrt{2}+7}, \Bigg(45927\,(b+4)\Bigg(\Bigg(\Bigg(-\frac{I\sqrt{2}\,b}{14} - \frac{3\,b}{28} - \frac{15}{14}\Bigg)\sqrt{1904\,I\sqrt{2}-5983} + \Bigg(I\,b^2+10\,I\,b+\frac{51\,I}{7}\Bigg)\sqrt{2} + \frac{17\,b^2}{56} + \frac{85\,b}{28} - 48\Bigg)\Bigg)\Bigg/\Bigg(\Bigg(4\,I\sqrt{2}+7\Bigg)^3\Bigg(4\,I\sqrt{2}-7\Bigg)^2\Bigg(56\,I\sqrt{2}\,b - 6\sqrt{1904\,I\sqrt{2}-5983} + 17\,b\Bigg)\Bigg), 1+\frac{b}{6},$$

$$\frac{251505\Bigg(\Bigg(-\frac{6\,I\sqrt{2}}{115}-\frac{21}{230}\Bigg)\sqrt{1904\,I\sqrt{2}-5983} + b\Bigg(I\sqrt{2}-\frac{329}{460}\Bigg)\Bigg)(12+b)}{(4\,I\sqrt{2}+7)^3\,(4\,I\sqrt{2}-7)^2\,(56\,I\sqrt{2}\,b-6\sqrt{1904\,I\sqrt{2}-5983}+17\,b)},$$

$$\frac{b}{4}+1, \frac{3}{2}, -\frac{27\,x}{4\,I\sqrt{2}+7}\Bigg)\, x^{\frac{3}{8}+\frac{b}{8}}\,(27\,x+4\,I\sqrt{2}+7)^{3/4}\,(-27\,x+4\,I\sqrt{2}-7)^{-\frac{6561\,(57113+10604\,I\sqrt{2})}{4\,(56\,I\sqrt{2}+17)^2\,(4\,I\sqrt{2}-7)^2\,(4\,I\sqrt{2}+7)^3}}\Bigg) \qquad (1.11)$$

$$+ \frac{1}{(27\,x^2+14\,x+3)^{1/4}} \Bigg(\_C2\, HG\Bigg(\frac{-4\,I\sqrt{2}+7}{4\,I\sqrt{2}+7}, \Bigg(1561518\,(b-4)\Bigg(\Bigg(\Bigg(\Bigg(-\frac{67\,I\,b}{476}+\frac{30\,I}{17}\Bigg)\sqrt{2} - \frac{275\,b}{952} + \frac{15}{28}\Bigg)\sqrt{1904\,I\sqrt{2}-5983} + \Bigg(I\,b^2 - 10\,I\,b$$

$$+ \frac{17949\,\mathrm{I}}{238}\Big)\sqrt{2} - \frac{5983\,b^2}{1904} + \frac{29915\,b}{952} + 48\Big)\Big) \Big/ \Big(\big((56\,\mathrm{I}\sqrt{2} + 17)\,(4\,\mathrm{I}\sqrt{2}$$

$$+ 7)^3\,(4\,\mathrm{I}\sqrt{2} - 7)^2\,\big(56\,\mathrm{I}\sqrt{2}\,b - 6\sqrt{1904\,\mathrm{I}\sqrt{2} - 5983} + 17\,b\big)\big)\Big),\ 1 - \frac{b}{12},$$

$$-\Big(503010\,\Big(\Big(-\frac{6\,\mathrm{I}\sqrt{2}}{115} - \frac{21}{230}\Big)\sqrt{1904\,\mathrm{I}\sqrt{2} - 5983} + b\,\Big(\mathrm{I}\sqrt{2} - \frac{329}{460}\Big)\Big)\,(-6$$

$$+ b)\Big) \Big/ \Big((4\,\mathrm{I}\sqrt{2} + 7)^3\,(4\,\mathrm{I}\sqrt{2} - 7)^2\,\big(56\,\mathrm{I}\sqrt{2}\,b - 6\sqrt{1904\,\mathrm{I}\sqrt{2} - 5983}$$

$$+ 17\,b\big)\Big),\ -\frac{b}{4} + 1,\ \frac{3}{2},\ -\frac{27\,x}{4\,\mathrm{I}\sqrt{2} + 7}\Big)\,x^{\frac{3}{8} - \frac{b}{8}}\,\big(27\,x + 4\,\mathrm{I}\sqrt{2} + 7\big)^{3/4}\,\big(-27\,x$$

$$+ 4\,\mathrm{I}\sqrt{2} - 7\big)^{-\frac{6561\,(57113 + 10604\,\mathrm{I}\sqrt{2})}{4\,(56\,\mathrm{I}\sqrt{2} + 17)^2\,(4\,\mathrm{I}\sqrt{2} - 7)^2\,(4\,\mathrm{I}\sqrt{2} + 7)^3}}\Bigg)$$

> $dsolve(ode_6, [hypergeometricsols])$

$$y(x) = \_C1\,\Big(1 + \frac{14}{3}\,x + 9\,x^2\Big)^{\frac{b}{24} - \frac{1}{8}}\,\Big(1 - \frac{x}{3}\Big)^{\frac{b}{12} - \frac{1}{4}}\,x^{\frac{3}{8} - \frac{b}{8}}\,{}_2F_1\Big(\frac{5}{8} - \frac{b}{24}, -\frac{b}{24} \tag{1.12}$$

$$+ \frac{1}{8}; 1 - \frac{b}{12}; -\frac{256\,x^3}{(x-3)^2\,(27\,x^2 + 14\,x + 3)}\Big) + \_C2\,\Big(1 + \frac{14}{3}\,x + 9\,x^2\Big)^{-\frac{b}{24}}$$

$$-\frac{1}{8}\,\Big(1 - \frac{x}{3}\Big)^{-\frac{b}{12} - \frac{1}{4}}\,x^{\frac{3}{8} + \frac{b}{8}}\,{}_2F_1\Big(\frac{5}{8} + \frac{b}{24}, \frac{b}{24} + \frac{1}{8}; 1 + \frac{b}{12};$$

$$-\frac{256\,x^3}{(x-3)^2\,(27\,x^2 + 14\,x + 3)}\Big)$$

> $odetest(\textbf{(1.12)}, ode_6)$

$$0 \tag{1.13}$$

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Linearization of polynomially nonlinear ODE and PDE

A fingerprint of a linear DE is that linear super-position of its solutions yields other solutions. Encoded geometrically this means that the input DE must have a subgroup of symmetries corresponding to this super-position symmetry (and in particular be an infinite abelian Lie subgroup in the case of linearizable PDE).

IsLinearizable and MapDE leverage the power of the LieAlgebrasOfVectorFields package, introduced in Maple 2020, by algorithmically computing and exploiting symmetry group properties through their associated Lie algebras. Lie symmetry groups, when linearized about the identity symmetry, yield Lie algebras of vector fields tangent to their one parameter group orbits.

This beautiful linearization yields determining systems of linear homogeneous PDE (LHPDE) for the components of the associated Lie algebra of vector fields.

Then algorithmic differential reduction and elimination algorithms are applied to these linear systems, to find both algebraic (e.g. center, lower central series, etc) and geometric (e.g. distribution, invariants, etc) properties of the Lie algebras of vector fields.

The LieAlgebrasOfVectorFields package is a complement to DifferentialGeometry. In particular, the LieAlgebras package in DifferentialGeometry enables the flexible formulation of geometric problems arising in applications, enabling the user to formulate and solve them using exterior differential systems and moving frames while working intrinsically on the relevant manifolds.

The IsLinearizable command illustrating the existence step of MapDE

The IsLinearizable command introduced in Maple 2020 illustrates some basic ideas underlying MapDE. In particular it uses LieAlgebrasOfVectorFields (LAVF) to implement a remarkable algorithm introduced by Lyakhov, Gerdt and Michels (2017).

To illustrate it, consider the nonlinear ODE:

```
> with(LieAlgebrasOfVectorFields) :
```

$$> NLODE := u(x)\left(\frac{d^3}{dx^3}u(x)\right) + 3\left(\frac{d}{dx}u(x)\right)\left(\frac{d^2}{dx^2}u(x)\right) - \frac{3\left(\left(\frac{d}{dx}u(x)\right)^2 + u(x)\left(\frac{d^2}{dx^2}u(x)\right) + 1\right)^2}{u(x)\left(\frac{d}{dx}u(x)\right) + x} - \frac{8x\left(u(x)\left(\frac{d}{dx}u(x)\right) + x\right)^4\left(1 + u(x)^2 + x^2\right)}{u(x)^2 + x^2} = 0 :$$

To check whether this ODE can be linearized you can use *IsLinearizable.* First, set up the symmetry vector field

```
> X := VectorField([[xi(x, u), x], [eta(x, u), u]], space = [x, u])
```

$$X := \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} \tag{2.1.1}$$

Then apply IsLinearizable

```
> IsLinearizable(NLODE, X)
```

true **(2.1.2)**

Here *true* means that there exists an invertible change of variables that maps the nonlinear ODE to a linear ODE. IsLinearizable is a good example of how LAVF enables the efficient computation of geometric information, in this case the existence of a linearization map, to determine if more expensive methods can be applied to a problem (in this case to determine the linearization). Here we display the simple procedure, which uses the LAVF commands SymmetryLAVF, SolutionDimension, DerivedAlgebra and IsAbelian.

```
> print(IsLinearizable)
```

(2.1.3)

```
proc(DEs::equation, X::LieAlgebrasOfVectorFields:-VectorField := NULL, $)
    local n, L, m, Da, S;
    if DEs::equation then S := [DEs] else S := DEs fi;
    n := PDEtools[difforder](S);
```

```
    if n < 2 then return true fi;
    try L := LieAlgebrasOfVectorFields:-SymmetryLAVF(S, X) catch: error  end;
    m := SolutionDimension(L);
    if n = 2 and m = 8 then return true fi;
    if m = n + 4 then return true fi;
    if m = n + 1 or m = n + 2 then
        Da := DerivedAlgebra(L); return evalb(IsAbelian(Da) and n
         = SolutionDimension(Da))
    fi;
    return false
end
```

In the procedure above, for our example, n = 3. SymmetryLAVF sets up the Lie algebra of vector fields, and m is the dimension of the Lie symmetry algebra m = 4 = n + 1 . The LAVF instruction IsAbelian(Da) yields true, without determining a basis for the Derived Algebra Da. Only algorithmic differentiations and eliminations of the determining system for Da were needed, based in an induced Lie algebra on the space of initial data for Da.

Apply MapDE to

$$u\, u_{x,x,x} + 3\, u_x\, u_{x,x} - \frac{3\left(u\, u_{x,x} + u_x^2 + 1\right)^2}{u\, u_x + x} - \frac{8\, x\left(u\, u_x + x\right)^4\left(u^2 + x^2 + 1\right)}{u^2 + x^2} = 0$$

The IsLinearizable command in the previous section only determines the existence of a mapping of an input DE to some (unknown) linear (target) DE. Here we illustrate how MapDE determines more information about the mapping, and in some cases can determine the map explicitly.

Applying MapDE to NLODE:

We set the MapDE infolevel to get some information about the main steps of the computation:

```
> infolevel[MapDE] := 1 :
```

```
> MLODE := MapDE([[NLODE], [[x], [u]], [[xi], [eta]]], [ToLinearDE, [[xh], [uh]]], [[psi],
    [phi]])
```

```
MapDE: Begin section where we compute sys S' of subalgebra L' of
L, containing solution of mapping problem.
MapDE: Single ODE so apply the Lyakhov-Gerdt-Michels ODE Test for
Existence of Linearization
MapDE: Input DE is linearizable by Lyakhov-Gerdt-Michels ODE
Test: dimL = 4
MapDE: Construct BK map eqns on Tangent Space to Jet Variety.
```

```
MapDE: Mapping equations constructed.
MapDE: Set up ranking and mindim specification for differential-
elimination.
MapDE: In ToLinear and not CK.
MapDE:  Settings for differential-elimination: MinDim = 3 ranking
= [rho[1] xi eta] RifStrategy = ezcriteria = [1 .9]
MapDE: Apply differential-elimination (rifsimp) to Mapping
Equations.
MapDE: Completed differential-elimination and casesplitting of
Mapping System. Time in rifsimp = 0.45e-1
MapDE: Existence of linearization was found earlier. Now
attempting to integrate mapping eqns
MapDE: Try to simplify PhiEqSol by setting values for the unknown
functions, constants etc.
MapDE: Attempted simplification of PhiEq yielded nonzero
Jacobian, now using simplified form
```

$$MLODE := \mathrm{table}\Bigg(\Bigg[IntegratedMap = \left[xh = u^2 + x^2, uh = -x\right], MapDEPivs = \left[\frac{\partial}{\partial u}\psi(x,u) \neq 0, \left(\frac{\partial}{\partial u}\phi(x,u)\right)x - \left(\frac{\partial}{\partial x}\phi(x,u)\right)u \neq 0\right], TargetDE = \left[\frac{d^3}{dxh^3}uh(xh) = -\frac{(xh+1)\,uh(xh)}{xh}\right] \textbf{where} \left[uh(xh)\left(\frac{d}{dxh}uh(xh)\right) \neq 0, -uh(xh)^2 + xh \neq 0, uh(xh) \neq 0\right], MapExists = true, RedMapDE = \left[\frac{\partial^2}{\partial x^2}\phi(x,u) = -\frac{1}{u^3}\left(\left(\frac{\partial^2}{\partial u^2}\phi(x,u)\right)u\,x^2 - 2\left(\frac{\partial^2}{\partial u\partial x}\phi(x,u)\right)u^2 x - \left(\frac{\partial}{\partial u}\phi(x,u)\right)u^2 - \left(\frac{\partial}{\partial u}\phi(x,u)\right)x^2\right), \frac{\partial}{\partial x}\psi(x,u) = \frac{\left(\frac{\partial}{\partial u}\psi(x,u)\right)x}{u}\right]\Bigg]\Bigg) \quad \textbf{(2.2.1)}$$

Here MapDE has carried out the existence step described in the previous section, to determine that there exists a mapping [xh = psi(x,u), uh = phi(x,u)] of NLODE to a linear ODE. Then MapDE is guaranteed in theory to return canonical differential equations in phi and psi for the mapping MLODE[RedMapDE] subject to the inequality constraints MLODE [MapDEPivs]. MapDE then uses pdsolve to attempt to integrate those equations to obtain an explicit form for the mapping, obtaining in this case:

> $MLODE[IntegratedMap]$

$$[xh = u^2 + x^2, uh = -x] \tag{2.2.2}$$

The target linear ODE is also obtained:

> $MLODE[TargetDE]$

$$\left[\frac{d^3}{dxh^3}\, uh(xh) = -\frac{(xh+1)\, uh(xh)}{xh}\right] \mathbf{where} \left[uh(xh)\left(\frac{d}{dxh}\, uh(xh)\right) \neq 0, -uh(xh)^2 + xh \neq 0, uh(xh) \neq 0\right] \tag{2.2.3}$$

Even though both NLODE and the considerably simpler linear MLODE can not be explicitly solved by dsolve, approximations for a basis of 3 solutions can be found by dsolve/series or dsolve/numeric. In this way the linearity of the target can be exploited to enable a general solution to be constructed without further significant computation.

Nonlinear PDE System linearizable by hodograph transformation $v_t = v^3 u_x, u_t = u^5 v_x$

> $infolevel[MapDE] := 2 :$

> $NPDESys := [diff(v(x, t), t) = v(x, t)^3\, diff(u(x, t), x), diff(u(x, t), t) = u(x, t)^5\, diff(v(x, t), x)]$

$$NPDESys := \left[\frac{\partial}{\partial t}\, v(x, t) = v(x, t)^3 \left(\frac{\partial}{\partial x}\, u(x, t)\right), \frac{\partial}{\partial t}\, u(x, t) = u(x, t)^5 \left(\frac{\partial}{\partial x}\, v(x, t)\right)\right] \tag{2.3.1}$$

We now compute the linearizing mapping and, when possible, also the resulting linear DE system, with *Map*DE

> $MPDESys_3 := MapDE([NPDESys, [[x, t], [u, v]], [[\xi, \tau], [\eta, \beta]]], [ToLinearDE, [[xh, th], [uh, vh]]], [[\psi, \Upsilon], [\varphi, \phi]])$

```
MapDE: Enter MapDE: Planned to have glossary of symbols and their meaning similar to rifsimp:
MapDE: ID = Initial Data
MapDE: Dim = Dimension
MapDE: DA  = Derived Algebra
MapDE: R  = input DE or its rif-form, Rh = Target DE or its rif-form
MapDE: S  = DetSys Sym of R, S' = DetSys for SubAlg of Sym of S
MapDE: Input DE of order 1 in 2 dependent variable(s) 2 independent variable(s) 2 equation(s)
MapDE: Enter Fast Early Algebraic-Geometric Prop Test
MapDE: Begin section where we compute sys S' of subalgebra L' of L, containing solution of mapping problem.
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MapDE: About to enter DerivedAlgebraInf
DerivedAlgebraInf: Enter DerivedAlgebraInf
DerivedAlgebraInf: Lie pseudogroup G of sym of DE: dim(G) =
infinity So DE is possibly linearizable
DerivedAlgebraInf: Apply rif to commutator DA sys
DerivedAlgebraInf: ID for DA sys =  [[] [xi(x[0] t[0] u[0] v) =
_F1(v) tau(x[0] t[0] u[0] v) = _F2(v)]]
MapDE: Exited from DerivedAlgebraInf. Time in DerivedAlgebraInf =
.588
MapDE: Construct BK map eqns on Tangent Space to Jet Variety.
MapDE: Mapping equations constructed.
MapDE: Set up ranking and mindim specification for differential-
elimination.
MapDE: In ToLinear and not CK.
MapDE:  Settings for differential-elimination: MinDim = infinity
ranking = [rho[1] rho[2] xi tau eta beta] RifStrategy =
ezcriteria = [1 .9]
MapDE:  For a detailed description of these settings see Maple's
documentation on rifsimp.
MapDE: Apply differential-elimination (rifsimp) to Mapping
Equations.
MapDE: Completed differential-elimination and casesplitting of
Mapping System. Time in rifsimp = 2.815
MapDE: After rif ncases = 46
MapDE: Number consistent cases satisfying minimum dim condns 2
MapDE: Multiple   2   cases found. Analyzing cases.
MapDE: nops(RifMapEqsSys)= 2 ntruecases= 2
MapDE: CaseSelectList = [2]
MapDE: Apply a pure elimination strategy to isolate the target
and prepare to find the transformations.
MapDE: PsiPhiEqs have triangular structure wrt phi - psi vars
MapDE: Existence of linearization was found earlier. Now
attempting to integrate mapping eqns
MapDE: Try to simplify PhiEqSol by setting values for the unknown
functions, constants etc.
MapDE: Attempted simplification of PhiEq yielded nonzero
Jacobian, now using simplified form
```

$$MPDESys_3 := \mathrm{table}\left(\left[IntegratedMap = [xh = u,\ th = v,\ uh = x,\ vh = t],\ MapDEPivs = \left[\frac{\partial}{\partial x}\ \phi(x, t, \right.\right.\right. \quad \textbf{(2.3.2)}$$

$$u, v) \neq 0, \frac{\partial}{\partial t} \varphi(x, t, u, v) \neq 0, \left(\frac{\partial}{\partial u} \psi(x, t, u, v)\right) \left(\frac{\partial}{\partial v} \Upsilon(x, t, u, v)\right) - \left(\frac{\partial}{\partial v} \psi(x, t, u,$$

$$v)\right) \left(\frac{\partial}{\partial u} \Upsilon(x, t, u, v)\right) \neq 0\Big], TargetDE = \left[\frac{\partial}{\partial xh} vh(xh, th) = \frac{\frac{\partial}{\partial th} uh(xh, th)}{xh^5}, \frac{\partial}{\partial th}\right.$$

$$vh(xh, th) = \frac{\frac{\partial}{\partial xh} uh(xh, th)}{th^3}, \frac{\partial^2}{\partial xh^2} uh(xh, th) = \left.\frac{\left(\frac{\partial^2}{\partial th^2} uh(xh, th)\right) th^3}{xh^5}\right] \textbf{where} \Big[$$

$$-\left(\frac{\partial}{\partial th} uh(xh, th)\right)^2 th^3 + \left(\frac{\partial}{\partial xh} uh(xh, th)\right)^2 xh^5 \neq 0, vh(xh, th) \neq 0\Big], MapExists$$

$$= true, RedMapDE = \left[\frac{\partial^2}{\partial x^2} \phi(x, t, u, v) = 0, \frac{\partial^2}{\partial x^2} \varphi(x, t, u, v) = 0, \frac{\partial^2}{\partial t \partial x} \varphi(x, t, u, v) = 0,\right.$$

$$\frac{\partial^2}{\partial t^2} \varphi(x, t, u, v) = 0, \frac{\partial}{\partial x} \Upsilon(x, t, u, v) = 0, \frac{\partial}{\partial x} \psi(x, t, u, v) = 0, \frac{\partial}{\partial t} \Upsilon(x, t, u, v) = 0, \frac{\partial}{\partial t}$$

$$\phi(x, t, u, v) = 0, \frac{\partial}{\partial t} \psi(x, t, u, v) = 0\Big]\Big]\Big)$$

Here MapDE finds an elegant hodograph form for the transformation to a linear system. Such hodograph transformations are important in applications, precisely because they linearize such problems. Notice that during the linearization analysis, MapDE detected highly non-trivial infinite dimensional Lie pseudogroups which were key to the analysis.

Fast certification of nonlinearizability of the KP Equation

$$\boldsymbol{u_{x,t} + \frac{3\,u_x^2}{2} + \frac{3\,u\,u_{x,x}}{2} + \frac{u_{x,x,x,x}}{4} + \frac{3\,u_{y,y}}{4} = 0}$$

Consider the KP (Kadomtsev–Petviashvili) equation which arises from nonlinear wave motion, and is a generalization of the KdV (Korteweg–de Vries) equation to two dimensions:

> $infolevel[MapDE] := 2$

$$infolevel_{MapDE} := 2 \tag{2.4.1}$$

> $KPDE := \left[diff\left(diff(u(x, y, t), t) + \frac{3}{2} u(x, y, t)\, diff(u(x, y, t), x) + \frac{1}{4} diff(u(x, y, t), x, x, x),\right.\right.$
$\left.\left. x\right) + \frac{3}{4} diff(u(x, y, t), y, y) = 0\right]$

$$KPDE := \left[\frac{\partial^2}{\partial t \partial x} u(x,y,t) + \frac{3 \left(\frac{\partial}{\partial x} u(x,y,t)\right)^2}{2} + \frac{3\, u(x,y,t) \left(\frac{\partial^2}{\partial x^2} u(x,y,t)\right)}{2} + \frac{\frac{\partial^4}{\partial x^4} u(x,y,t)}{4} + \frac{3 \frac{\partial^2}{\partial y^2} u(x,y,t)}{4} = 0 \right] \quad (2.4.2)$$

> $MKPDE := MapDE([KPDE, [[x, y, t], [u]], [[\xi, \eta, \tau], [\beta]]], [ToLinearDE, [[xh, yh, th], [uh]]], [[\psi, \Upsilon, \varphi], [\phi]])$

```
MapDE: Enter MapDE: Planned to have glossary of symbols and their
meaning similar to rifsimp:
MapDE: ID = Initial Data
MapDE: Dim = Dimension
MapDE: DA  = Derived Algebra
MapDE: R  = input DE or its rif-form, Rh = Target DE or its rif-
form
MapDE: S  = DetSys Sym of R, S' = DetSys for SubAlg of Sym of S
MapDE: Input DE of order 4 in 1 dependent variable(s) 3
independent variable(s) 1 equation(s)
MapDE: Enter Fast Early Algebraic-Geometric Prop Test
MapDE: Input R not linearizable since DiffDim(L}= 1  <  2  =
DiffDim(R))
```

$$MKPDE := \mathrm{table}([MapExists = false]) \quad (2.4.3)$$

Although the KP equation has an infinite dimensional Lie pseudogroup of symmetries it is not linearizable, since the differential dimension (a measure of size of such pseudogroups) is not large enough (not equal to the differential dimension of solutions of the KP equation)
.

To solve the equation in cases like this one, note that despite the non-existence of a *linearizing* mapping, the underlying point symmetries of the equation can be used to compute invariant solutions. The existence of point symmetries is guaranteed when the PDEtools:-DeterminingPDE command returns related infinitesimals different from zero

> $PDEtools\text{:-}DeterminingPDE(KPDE)$

$$\left\{ \frac{\partial^2}{\partial y^2} \_\xi_y(x,y,t,u) = 0, \frac{\partial}{\partial t} \_\xi_t(x,y,t,u) = \frac{3 \frac{\partial}{\partial y} \_\xi_y(x,y,t,u)}{2}, \frac{\partial}{\partial u} \_\xi_t(x,y,t,u) = 0, \frac{\partial}{\partial x} \_\xi_t(x,y,t,u) = 0, \frac{\partial}{\partial y} \_\xi_t(x,y,t,u) = 0, \frac{\partial}{\partial u} \_\xi_x(x,y,t,u) = 0, \frac{\partial}{\partial x} \_\xi_x(x,y,t,u) \right. \quad (2.4.4)$$

$$= \frac{\frac{\partial}{\partial y} \_\xi_y(x, y, t, u)}{2}, \frac{\partial}{\partial y} \_\xi_x(x, y, t, u) = -\frac{2 \frac{\partial}{\partial t} \_\xi_y(x, y, t, u)}{3}, \frac{\partial}{\partial u} \_\xi_y(x, y, t, u) = 0, \frac{\partial}{\partial x} \_\xi_y(x, y, t, u) = 0, \_\eta_u(x, y, t, u) = -u \left(\frac{\partial}{\partial y} \_\xi_y(x, y, t, u) \right) + \frac{2 \frac{\partial}{\partial t} \_\xi_x(x, y, t, u)}{3} \right\}$$

Indeed, in this example, infinitesimals depending on three arbitrary functions are computable using PDEtools:-Infinitesimals

> *PDEtools:-Infinitesimals(KPDE)*

$$\left[\_\xi_x(x, y, t, u) = \frac{\left(\frac{d}{dt} \_F1(t)\right) x}{2} - \frac{\left(\frac{d^2}{dt^2} \_F1(t)\right) y^2}{3} - \frac{2 \left(\frac{d}{dt} \_F2(t)\right) y}{3} + \_F3(t), \right. \tag{2.4.5}$$

$$\_\xi_y(x, y, t, u) = \left(\frac{d}{dt} \_F1(t)\right) y + \_F2(t), \_\xi_t(x, y, t, u) = \frac{3 \_F1(t)}{2} + 1, \_\eta_u(x, y, t, u) = -u \left(\frac{d}{dt} \_F1(t)\right) + \frac{\left(\frac{d^2}{dt^2} \_F1(t)\right) x}{3} - \frac{2 \left(\frac{d^3}{dt^3} \_F1(t)\right) y^2}{9} - \frac{4 \left(\frac{d^2}{dt^2} \_F2(t)\right) y}{9} + \frac{2 \frac{d}{dt} \_F3(t)}{3} \right]$$

With the purpose of using these infinitesimals to compute solutions, this general form **(2.4.5)** can be specialized

> *PDEtools:-Infinitesimals(KPDE, specialize_Fn)*

$$[\_\xi_x(x, y, t, u) = 0, \_\xi_y(x, y, t, u) = 1, \_\xi_t(x, y, t, u) = 1, \_\eta_u(x, y, t, u) = 0], [\_\xi_x(x, y, t, u) = 1, \_\xi_y(x, y, t, u) = 0, \_\xi_t(x, y, t, u) = 1, \_\eta_u(x, y, t, u) = 0], \left[\_\xi_x(x, y, t, u) = 0, \_\xi_y(x, y, t, u) = 0, \_\xi_t(x, y, t, u) = \frac{5}{2}, \_\eta_u(x, y, t, u) = 0\right], \left[\_\xi_x(x, y, t, u) = t, \_\xi_y(x, y, t, u) = 0, \_\xi_t(x, y, t, u) = 1, \_\eta_u(x, y, t, u) = \frac{2}{3}\right], \left[\_\xi_x(x, y, t, u) = -\frac{2 y}{3}, \_\xi_y(x, y, t, u) = t, \_\xi_t(x, y, t, u) = 1, \_\eta_u(x, y, t, u) = 0\right], \left[\_\xi_x(x, y, t, u) = \frac{x}{2}, \_\xi_y(x, y, t, u) = y, \_\xi_t(x, y, t, u) = \frac{3 t}{2} + 1, \_\eta_u(x, y, t, u) = -u\right] \tag{2.4.6}$$

Using any of the infinitesimals above, for example the first one, we get a solution

> *PDEtools:-InvariantSolutions(KPDE,* **(2.4.6)**[1])

$$\left\{u(x, y, t) = \frac{\left(-4\_C2^4 - 4\_C2\_C3 - 3\_C3^2\right)\cosh(\_C2\,x + \_C3\,(-y + t) + \_C1)^2 + 12\_C2^4}{6\cosh(\_C2\,x + \_C3\,(-y + t) + \_C1)^2\_C2^2}\right\} \tag{2.4.7}$$

From the form of this solution one can infer this equation also admits travelling wave solutions

> $PDEtools{:}\text{-}TWSolutions(KPDE)$

$$\{u(x, y, t) = \_C5\}, \left\{u(x, y, t) = -2\_C2^2\tanh(\_C2\,x + \_C3\,y + \_C4\,t + \_C1)^2 + \frac{8\_C2^4 - 4\_C2\_C4 - 3\_C3^2}{6\_C2^2}\right\} \tag{2.4.8}$$

References

(1) D.A. Lyakhov, V.P. Gerdt, and D.L. Michels. *Algorithmic Verification of Linearizability for Ordinary Differential Equations.* Proceedings of ISSAC 2017. ACM Press, 2017.

(2) Z. Mohammadi, G. Reid, and S.-L.T Huang. *Introduction of the MapDE algorithm for determination of mappings relating differential equations.* Proceedings of ISSAC 2019, pp. 331–338. ACM Press, 2019.

(3) Z. Mohammadi, G. Reid, and S.-L.T. Huang. *Symmetry-based algorithms for invertible mappings of polynomially nonlinear PDE to linear PDE.* Mathematics in Computer Science, 1–24, 2020.